

ISOPERIMETRIC EQUALITIES FOR ROSETTES

MICHAŁ ZWIERZYŃSKI

ABSTRACT. In this paper we study the isoperimetric-type equalities for rosettes, i.e. regular closed planar curves with non-vanishing curvature. We find the exact relations between the length and the oriented area of rosettes based on the oriented areas of the Wigner caustic, the Constant Width Measure Set and the Spherical Measure Set.

We also study and find new results about the geometry of affine equidistants of rosettes and of the union of rosettes.

1. INTRODUCTION

The *classical isoperimetric inequality* for plane curves states the following.

Theorem 1.1. (*Isoperimetric inequality*) *Let M be a simple curve. Let L_M , A_M be the length of M and the area of region enclosed by M , respectively. Then*

$$L_M^2 \geq 4\pi A_M,$$

and the equality holds if and only if M is a circle.

This result is known already in the Ancient Greece but first proof was given in the nineteenth century by Steiner [34]. After this paper, there have been many new proofs, applications and generalizations of this theorem, see [2, 11, 12, 13, 19, 20, 22, 26, 34, 37, 38] and the literature therein. Recently in [37, 38] we prove the improved isoperimetric-type inequalities and equalities for planar ovals.

Theorem 1.2. [37, 38] (*Isoperimetric-type equalities*) *Let M be an oval. Then*

$$L_M^2 = 4\pi A_M + 8\pi \left| \tilde{A}_{E_{0.5}(M)} \right| + \pi \left| \tilde{A}_{CWMS(M)} \right|,$$

and if M is not a curve of constant width, then

$$L_M^2 = 4\pi A_M + 4\pi \left| \tilde{A}_{SMS(M)} \right|,$$

where $\tilde{A}_{E_{0.5}(M)}$, $\tilde{A}_{CWMS(M)}$, $\tilde{A}_{SMS(M)}$ are the oriented area of the Wigner caustic of M , the oriented area of the Constant Width Measure Set of M the oriented area of the Spherical Measure Set of M , respectively.

Furthermore M is a curve of constant width if and only if

$$L_M^2 = 4\pi A_M + 8\pi \left| \tilde{A}_{E_{0.5}(M)} \right|,$$

and M is a curve which has the center of symmetry if and only if

$$L_M^2 = 4\pi A_M + \pi \left| \tilde{A}_{CWMS(M)} \right|.$$

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The *Wigner caustic* of an oval on the affine symplectic plane was first introduced by Berry, in his celebrated 1977 paper [1] on the semiclassical limit of Wigner's phase-space representation of quantum states. The Wigner caustic is an example of an *affine λ -equidistant* (for $\lambda = \frac{1}{2}$). The affine λ -equidistant is the set of points divided chords connecting points on M where tangent lines to M are parallel in the ratio λ . There are many papers considering affine equidistants, see [4, 6, 7, 8, 9, 14, 16, 24, 32, 36]. In [4, 14] the Wigner caustic is known as the *area evolute* and in [24] is known as the *symmetry defect*. The well-known Blaschke-Süss theorem ([18, 27]) states that if M is an oval then M has at least three *antipodal pairs*, i.e. two points where the tangent lines at these points are parallel and the curvatures are equal. Singularities of the Wigner caustic occur exactly from an antipodal pair of an oval. The Wigner caustic also leads to one of two constructions of improper affine spheres ([5]). In [9] we study the global geometry of the Wigner caustic and equidistants of plane curves, we find many global results, for instance we presented an algorithm to describe smooth branches of equidistants of parametrized closed planar curves. By this algorithm we can find among other things the exact number of smooth branches, the number of inflexion points, the parity of the number of cusps of each branch. In this paper we use this method in Section 3.

The paper is organized as follows.

In Section 2 we study the geometry of affine λ -equidistants, the Wigner caustic, the Constant Width Measure Set and the Spherical Measure Set of rosettes, i.e. closed smooth regular curves with non-vanishing curvature. We prove generalizations of Theorem 1.2 for rosettes.

Section 3 contains theorems on the geometry of branches of affine λ -equidistants of the union of rosettes.

2. ROSETTES

Rosettes were studied in details in many papers, see [3, 28, 29, 30] and the literature therein. In particular well studied ovals are examples of rosettes.

Definition 2.1. Let m be a positive integer. An *m -rosette* is a regular closed curve with non-vanishing curvature of the rotation number equal to m .

Remark 2.2. By the above definition we get also that every rosette is positively oriented.

Let R_m be an m -rosette. Let M be a regular closed curve.

Definition 2.3. A pair of points $a, b \in M$ ($a \neq b$) is called a *parallel pair* if the tangent lines to M at a and b are parallel.

Definition 2.4. An *affine λ -equidistant* is the following set

$$E_\lambda(M) = \{ \lambda a + (1 - \lambda)b \mid a, b \text{ is a parallel pair of } M \}.$$

The set $E_{0.5}(M)$ will be called the *Wigner caustic* of M .

Note that, for any given $\lambda \in \mathbb{R}$, we have $E_\lambda(M) = E_{1-\lambda}(M)$. Thus, the case $\lambda = \frac{1}{2}$ is special. In particular we have $E_0(M) = E_1(M) = M$.

If M is a generic closed regular curve then $E_\lambda(M)$ for $\lambda \neq 0, 1$ is a union of smooth parametrized curves. Each of these curves we will call a *smooth branch* of $E_\lambda(M)$. Generically each such branch has at most cusp singularities ([1, 17]).

In Fig. 1 we present (i) a rosette R_4 , (ii) $E_{0.4}(R_4)$, (iii) $E_{0.5}(R_4)$ and in (iv-viii) different smooth branches of $E_{0.5}(R_4)$.

Definition 2.5. A smooth curve \mathcal{L} is a *convex loop* if \mathcal{L} is parametrized by a smooth function $f : [0, b] \rightarrow \mathbb{R}^2$ such that the region bounded by \mathcal{L} is strictly convex, $f(0) = f(b)$ and vectors $f'(0)$, $f'(b)$ are linearly independent.

In [9] we study the geometry of E_λ of rosettes and obtain the following result.

Theorem 2.6. [9] *Let R_m be a generic rosette. Then*

- (i) *the number of smooth branches of $E_{0.5}(R_m)$ is equal to m ,*
- (ii) *at least $\lfloor \frac{m}{2} \rfloor$ branches of $E_{0.5}(R_m)$ are rosettes,*
- (iii) *$m-1$ branches of $E_{0.5}(R_m)$ have a rotation number equal to m and one branch has a rotation number equal to $\frac{m}{2}$,*
- (iv) *every smooth branch of $E_{0.5}(R_m)$ has an even number of cusps if m is even,*
- (v) *exactly one branch of $E_{0.5}(R_m)$ has an odd number of cusps if m is odd,*
- (vi) *cusps of $E_{0.5}(R_m)$ created from convex loops of M are in the same smooth branch of $E_{0.5}(R_m)$,*
- (vii) *the total number of cusps of $E_{0.5}(R_m)$ is not smaller than 2,*
- (viii) *the number of smooth branches of $E_\lambda(R_m)$ for $\lambda \neq 0, 0.5, 1$ is equal to $2m-1$,*
- (ix) *the rotation number of every smooth branch of $E_\lambda(R_m)$ for $\lambda \neq 0, 0.5, 1$ is equal to m ,*
- (x) *at least $m-1$ branches of $E_\lambda(R_m)$ for $\lambda \in (0, 0.5) \cup (0.5, 1)$ are rosettes,*
- (xi) *at least m branches of $E_\lambda(R_m)$ for $\lambda \in (-\infty, 0) \cup (1, \infty)$ are rosettes.*

In this section we improve Theorem 2.6 - see Theorem 2.8, Theorem 2.9 and Theorem 2.17.

Let us consider an oval R_1 . Take a point Θ as the origin of our frame. Let p be the oriented perpendicular distance from Θ to the tangent line at a point on R_1 . Let θ be the oriented angle from the positive x_1 -axis to this perpendicular ray. Clearly, p is a 2π -periodic function. It is well known (see for instance [20, 23, 33]) that a parameterization of R_1 in terms of $\theta, p(\theta)$ is as follows

$$(2.1) \quad \gamma_{R_1}(\theta) = (\gamma_{R_1,1}(\theta), \gamma_{R_1,2}(\theta)) = (p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta).$$

The pair $\theta, p(\theta)$ is known as the *polar tangential coordinate* on R_1 and $p(\theta)$ its Minkowski's support function.

Then, the curvature κ of R_1 is in the following form

$$(2.2) \quad \kappa(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0,$$

or equivalently, the radius of a curvature ρ of R_1 is given by

$$(2.3) \quad \rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta).$$

Let L_{R_1} and A_{R_1} be the length of R_1 and the area it bounds, respectively. Then one can get that

$$(2.4) \quad L_{R_1} = \int_{R_1} ds = \int_0^{2\pi} \rho(\theta) d\theta = \int_0^{2\pi} p(\theta) d\theta,$$

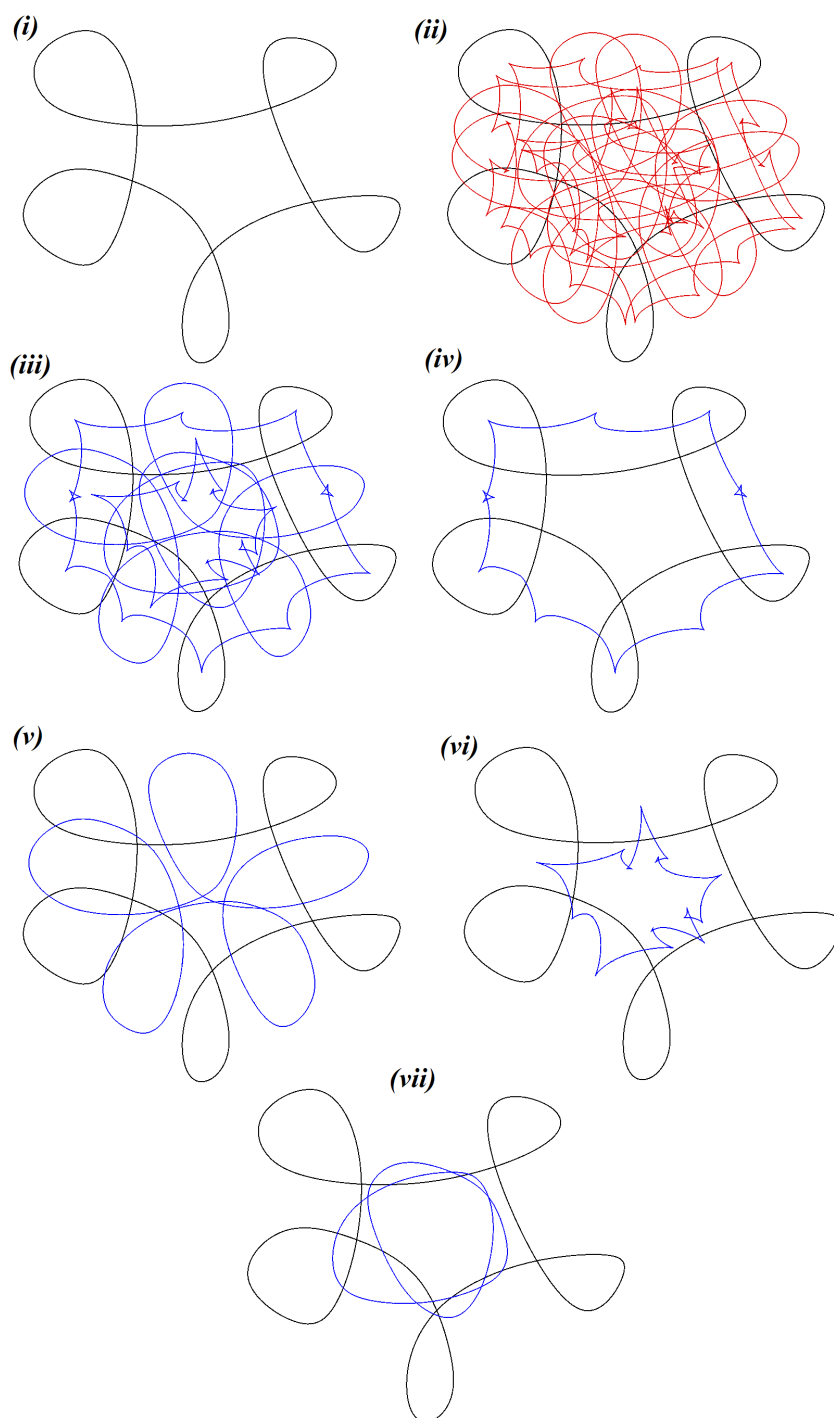


FIGURE 1.

and

$$(2.5) \quad \begin{aligned} A_{R_1} &= \frac{1}{2} \int_{R_1} p(\theta) ds \\ &= \frac{1}{2} \int_0^{2\pi} p(\theta) [p(\theta) + p''(\theta)] d\theta = \frac{1}{2} \int_0^{2\pi} [p^2(\theta) - p'^2(\theta)] d\theta. \end{aligned}$$

(2.4) and (2.5) are known as *Cauchy's formula* and *Blaschke's formula*, respectively.

The notion of support function can be generalized to obtain function of similar properties, but for rosettes ([28, 33]), but in this case the support function of R_m is $2m\pi$ -periodic.

Let p be the support function of R_m and γ be the parameterization of R_m .

Let $E_{0.5,k}(R_m)$ for $k = 1, 2, \dots, m$ be different branches of the Wigner caustic of R_m . Then by [9] the parameterization of $E_{0.5,k}(R_m)$ is as follows.

$$(2.6) \quad \gamma_{0.5,k}(\theta) = \frac{1}{2}(\gamma(\theta) + \gamma(\theta + k\pi)).$$

Remark 2.7. The map $R_m \ni \gamma(\theta) \mapsto \gamma_{0.5,m}(\theta) \in E_{0.5,m}(R_m)$ is the double covering of $E_{0.5,m}(R_m)$.

Then the support function of $E_{0.5,k}(R_m)$ is in the following form.

$$(2.7) \quad p_{\lambda,k}(\theta) = \lambda p(\theta) + (-1)^k (1 - \lambda) p(\theta + k\pi).$$

In Fig. 1(iv) there is $E_{0.5,1}(R_4)$, in (v) $E_{0.5,2}(R_4)$, in (vi) $E_{0.5,3}(R_4)$ and in (vii) $E_{0.5,4}(R_4)$.

Theorem 2.8. (i) If k is even and $k < m$ then

$$L_{E_{0.5,k}(R_m)} = L_{R_m}.$$

(ii) If k is even and $k = m$ then

$$2L_{E_{0.5,k}(R_m)} = L_{R_m}.$$

(iii) If k is odd and $k < m$ then

$$L_{E_{0.5,k}(R_m)} \leq L_{R_m}.$$

(iv) If k is odd and $k = m$ then

$$2L_{E_{0.5,k}(R_m)} \leq L_{R_m}.$$

(v) If k is even and $\lambda \in (0, 1) - \{0.5\}$, then

$$L_{E_{\lambda,k}(R_m)} = L_{R_m}$$

(vi) If k is odd and $\lambda \in (0, 1) - \{0.5\}$, then

$$L_{E_{\lambda,k}(R_m)} \leq L_{R_m}$$

(vii) If k is odd and $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$L_{E_{\lambda,k}(R_m)} = L_{R_m}$$

(viii) If k is even and $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$L_{E_{\lambda,k}(R_m)} \leq (|\lambda| + |1 - \lambda|) L_{R_m}$$

Proof. We prove (i–iv), other points are analogous.

Let us notice that $\rho_{0.5,k}(\theta) = \frac{1}{2}|\rho(\theta) + (-1)^k \rho(\theta + k\pi)|$.

(i) If k is even and $k < m$, then

$$\begin{aligned} L_{E_{0.5,k}(R_m)} &= \int_0^{2m\pi} \rho_{0.5,k}(\theta) d\theta \\ &= \frac{1}{2} \int_0^{2m\pi} (\rho(\theta) + \rho(\theta + k\pi)) d\theta \\ &= \frac{1}{2} (L_{R_m} + L_{R_m}) = L_{R_m}. \end{aligned}$$

(ii) It is a consequence of (i) and Remark 2.7.

(iii) If k is odd and $k < m$, then

$$\begin{aligned} L_{E_{0.5,k}(R_m)} &= \int_0^{2m\pi} \rho_{0.5,k}(\theta) d\theta \\ &= \frac{1}{2} \int_0^{2m\pi} |\rho(\theta) - \rho(\theta + k\pi)| d\theta \\ &\leq \frac{1}{2} (L_{R_m} + L_{R_m}) = L_{R_m}. \end{aligned}$$

(iv) It is a consequence of (iii) and Remark 2.7. □

The support function p of R_m has the following Fourier series.

$$(2.8) \quad p(\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right).$$

Therefore we get the following equalities.

$$(2.9) \quad p'(\theta) = \frac{1}{m} \sum_{n=1}^{\infty} n \left(-a_n \sin \frac{n\theta}{m} + b_n \cos \frac{n\theta}{m} \right).$$

(2.10)

$$p''(\theta) = -\frac{1}{m^2} \sum_{n=1}^{\infty} n^2 \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right).$$

$$(2.11) \quad \rho(\theta) = p(\theta) + p''(\theta) = a_0 - \frac{1}{m^2} \sum_{n=1}^{\infty} (n^2 - m^2) \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right).$$

$$(2.12) \quad L_{R_m} = \int_0^{2m\pi} |\gamma'(\theta)| d\theta = \int_0^{2m\pi} \rho(\theta) d\theta = \int_0^{2\pi} p(\theta) d\theta = 2\pi m a_0.$$

$$(2.13) \quad \tilde{A}_{R_m} = \frac{1}{2} \int_0^{2m\pi} (p^2(\theta) - p'^2(\theta)) d\theta = m\pi a_0^2 - \frac{\pi}{2m} \sum_{n=1}^{\infty} (n^2 - m^2) (a_n^2 + b_n^2).$$

Theorem 2.9. *Let R_m be a generic rosette. Then exactly $\left\lfloor \frac{m}{2} \right\rfloor$ branches of $E_{0.5}(R_m)$ are rosettes.*

Proof. By Theorem 6.1. in [9] we know that branches $E_{0.5,2}(R_m)$, $E_{0.5,4}(R_m)$, \dots , $E_{0.5,2 \cdot \lfloor 0.5m \rfloor}(R_m)$ are rosettes. We will show that other branches has at least one

singular point. It is enough to show that there exists θ such that $\rho_{0.5,k}(\theta) = 0$ for $k = 1, 3, \dots, 2 \left\lceil \frac{m}{2} \right\rceil - 1$. By (2.7) we get that

$$\begin{aligned}\rho_{0.5,k}(\theta) &= 0, \\ p_{0.5,k}(\theta) + p''_{0.5,k}(\theta) &= 0, \\ \rho(\theta) - \rho(\theta + k\pi) &= 0.\end{aligned}$$

So the number of singular points of $E_{0.5,k}(R_m)$ is equal to the number of zeros of $f(\theta) = \rho(\theta) - \rho(\theta + k\pi)$ in the range $[0, 2m\pi)$. Since $f(0) = f(2m\pi)$ and $\int_0^{2m\pi} f(\theta) d\theta = 0$, therefore there are at least two zeros of $f(\theta)$ in $[0, 2m\pi)$. \square

The lower bound of 2 cusps of the branches of the Wigner caustic which are not rosettes can be obtained, see Fig. 2.

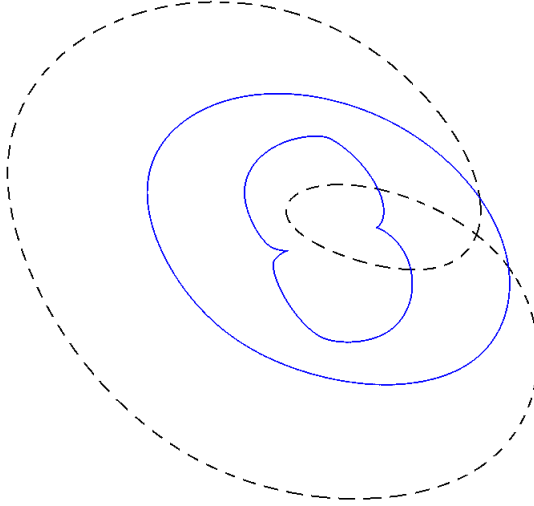


FIGURE 2. A 2-rosette (the dashed line) and two branches of the Wigner caustic.

Corollary 2.10. *A rosette R_m has at least $2 \left\lceil \frac{m}{2} \right\rceil$ antipodal pairs.*

Lemma 2.11. *The oriented area of $E_{0.5,k}(R_m)$ for $k < m$ in terms of coefficients of the Fourier series of the Minkowski support function p of R_m is equal*

(2.14)

$$\tilde{A}_{E_{0.5,k}(R_m)} = \frac{1 + (-1)^k}{2} m\pi a_0^2 - \frac{\pi}{4m} \sum_{n=1}^{\infty} (n^2 - m^2) \left(1 + (-1)^k \cos \frac{nk\pi}{m} \right) (a_n^2 + b_n^2),$$

and for $k = m$

(2.15)

$$\tilde{A}_{E_{0.5,m}(R_m)} = \frac{1 + (-1)^m}{4} m\pi a_0^2 - \frac{\pi}{8m} \sum_{n=1}^{\infty} (n^2 - m^2) (1 + (-1)^{n+m}) (a_n^2 + b_n^2).$$

Proof. By (2.7) and (2.13) we get the following equalities for $k < m$.

$$\begin{aligned} \tilde{A}_{E_{0.5,k}(R_m)} &= \frac{1}{2} \int_0^{2m\pi} \left(p_{0.5,k}^2(\theta) - p_{0.5,k}'^2(\theta) \right) d\theta \\ &= \frac{1}{8} \int_0^{2m\pi} \left[\left(p(\theta) + (-1)^k p(\theta + k\pi) \right)^2 - \left(p'(\theta) + (-1)^k p'(\theta + k\pi) \right)^2 \right] d\theta \\ &= \frac{1}{2} \tilde{A}_{R_m} + \frac{(-1)^k}{2} \Psi_{R_m}, \end{aligned}$$

where

$$(2.16) \quad \Psi_{R_m} = \frac{1}{2} \int_0^{2m\pi} \left(p(\theta)p(\theta + k\pi) - p'(\theta)p'(\theta + k\pi) \right) d\theta.$$

By (2.8) we get the following:

$$(2.17) \quad p(\theta + k\pi) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \left(\frac{n\theta}{m} + \frac{nk\pi}{m} \right) + b_n \sin \left(\frac{n\theta}{m} + \frac{nk\pi}{m} \right) \right] d\theta.$$

One can check that

$$(2.18) \quad \int_0^{2m\pi} \cos \frac{n_1\theta}{m} \cos \left(\frac{n_2\theta}{m} + \frac{n_2k\pi}{m} \right) d\theta = m\pi \cos \frac{n_2k\pi}{m} \cdot \delta_{n_1,n_2},$$

$$(2.19) \quad \int_0^{2m\pi} \sin \frac{n_1\theta}{m} \sin \left(\frac{n_2\theta}{m} + \frac{n_2k\pi}{m} \right) d\theta = m\pi \cos \frac{n_2k\pi}{m} \cdot \delta_{n_1,n_2},$$

$$(2.20) \quad \int_0^{2m\pi} \cos \frac{n_1\theta}{m} \sin \left(\frac{n_2\theta}{m} + \frac{n_2k\pi}{m} \right) d\theta = m\pi \sin \frac{n_2k\pi}{m} \cdot \delta_{n_1,n_2},$$

$$(2.21) \quad \int_0^{2m\pi} \sin \frac{n_1\theta}{m} \cos \left(\frac{n_2\theta}{m} + \frac{n_2k\pi}{m} \right) d\theta = -m\pi \sin \frac{n_2k\pi}{m} \cdot \delta_{n_1,n_2},$$

where δ_{n_1,n_2} is the Kronecker delta, i.e. $\delta_{n_1,n_2} = \begin{cases} 1, & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases}$.

Therefore by (2.18–2.21) we can express Ψ_{R_m} in the following way.

$$\begin{aligned} (2.22) \quad \Psi_{R_m} &= \frac{1}{2} \int_0^{2m\pi} \left(p(\theta)p(\theta + k\pi) - p'(\theta)p'(\theta + k\pi) \right) d\theta \\ &= m\pi a_0^2 - \frac{\pi}{2m} \sum_{n=1}^{\infty} (n^2 - m^2) \cos \frac{nk\pi}{m} (a_n^2 + b_n^2). \end{aligned}$$

By (2.13) and (2.22) we get (2.14) and by Remark 2.7 we get (2.15)

□

Theorem 2.12. (*Corollary 5.1 in [28]*) *Each rosette of constant width of the rotation number equal to m is given by the following support function.*

$$(2.23) \quad p_{R_m}(\theta) = a_0 + \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right).$$

Theorem 2.13. *A rosette R_m is a rosette of constant width if and only if*

$$(2.24) \quad L_{R_m}^2 = 8\pi m \tilde{A}_{E_{0.5,m}(R_m)} + 2\pi m(1 - (-1)^m) \tilde{A}_{R_m}.$$

Proof. By (2.23) and by (2.13) we get that the oriented area of R_m is in the following form.

$$\tilde{A}_{R_m} = m\pi a_0^2 - \frac{\pi}{2m} \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} (n^2 - m^2)(a_n^2 + b_n^2).$$

Therefore by Lemma 2.11 and by (2.12) we get the following equalities.

$$\begin{aligned} 2\tilde{A}_{E_{0.5,m}(R_m)} &= \frac{1 + (-1)^m}{2} m\pi a_0^2 - \frac{\pi}{4m} \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} (n^2 - m^2)(1 - (-1)^m)(a_n^2 + b_n^2) \\ &= \frac{1 + (-1)^m}{2} m\pi a_0^2 - \frac{1 - (-1)^m}{2} \left(\tilde{A}_{R_m} - m\pi a_0^2 \right) \\ &= \pi m a_0^2 - \frac{1 - (-1)^m}{2} \tilde{A}_{R_m} \\ &= \frac{L_{R_m}^2}{4\pi m} - \frac{1 - (-1)^m}{2} \tilde{A}_{R_m}. \end{aligned}$$

Therefore we get (2.24). □

In [38] we introduced the following set for 1-rosettes.

Definition 2.14. The *Constant Width Measure Set* of R_1 is the following set

$$\text{CWMS}(R_1) = \left\{ a - b + \frac{L_{R_1}}{\pi} \cdot \mathbf{n}(a) \mid a, b \text{ is a parallel pair of } R_1 \right\},$$

where \mathbf{n} is the unit normal vector field compatible with the orientation of R_1 . We treat $\text{CWMS}(R_1)$ as a subset of a vector space $V = \mathbb{R}^2$. To visualize the CWMS and the other sets in the same picture let us assume that the origin of V is Θ .

We generalize the CWMS for all rosettes in the following way.

Definition 2.15. Let γ be parameterization of R_m in terms of θ and $p(\theta)$. Then the *Constant Width Measure Set* of R_m is a curve defined by the following parameterization

$$\gamma_{\text{CWMS}(R_m)}(\theta) = \gamma(\theta) - \gamma(\theta + m\pi) + \frac{L_{R_m}}{m\pi} \cdot \mathbf{n}(\theta).$$

It is easy to see that $\text{CWMS}(R_m)$ has the following support function

$$(2.25) \quad p_{\text{CWMS}(R_m)}(\theta) = p(\theta) - (-1)^m p(\theta + m\pi) - \frac{L_{R_m}}{m\pi}$$

and in terms of coefficients of the Fourier series of p

$$(2.26) \quad p_{\text{CWMS}(R_m)}(\theta) = (-1 - (-1)^m) a_0 + \sum_{n=1}^{\infty} (1 - (-1)^{n+m}) \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right).$$

Theorem 2.16. *Let R_m be a generic m -rosette. If m is odd then $CWMS(R_m)$ is a smooth curve with only cusp singularities and the number of cusp singularities of $CWMS(R_m)$ is even and positive.*

Proof. $CWMS(R_m)$ is a connected smooth curve. The set $CWMS(R_m)$ is singular if and only if

$$(2.27) \quad \rho_{CWMS(R_m)}(\theta) = 0.$$

By theory of Thom (1975) [35] one can get that generically $CWMS(M)$ has cusp singularity when (2.27) holds.

If m is odd the condition (2.27) is equivalent to

$$(2.28) \quad \rho(\theta) + \rho(\theta + m\pi) = \frac{L_{R_m}}{m\pi}.$$

The number of cusp singularities of $CWMS(R_m)$ is the number of zeros of (2.28) in the range $[0, 2m\pi)$. Since the function $\rho(\theta) + \rho(\theta + m\pi)$ is $m\pi$ -periodic, we get that the number of cusp singularities of $CWMS(R_m)$ is even. Let us assume that this number is equal to zero. Then we get that for all $\theta \in [0, m\pi)$ one of two following inequalities holds.

$$\begin{aligned} \rho(\theta) + \rho(\theta + m\pi) &< \frac{L_{R_m}}{m\pi}, \\ \rho(\theta) + \rho(\theta + m\pi) &> \frac{L_{R_m}}{m\pi}. \end{aligned}$$

If $\rho(\theta) + \rho(\theta + m\pi) < \frac{L_{R_m}}{m\pi}$ for all $\theta \in [0, m\pi)$, then

$$\begin{aligned} \int_0^{m\pi} (\rho(\theta) + \rho(\theta + m\pi)) d\theta &< \int_0^{m\pi} \frac{L_{R_m}}{m\pi} d\theta, \\ L_{R_m} &< L_{R_m}. \end{aligned}$$

Since we get the same contradiction for $\rho(\theta) + \rho(\theta + m\pi) > \frac{L_{R_m}}{m\pi}$, the number of cusp singularities of $CWMS(R_m)$ is positive. □

In Fig. 3 we present examples of $CWMS(R_m)$.

Theorem 2.17. *(Isoperimetric equalities for rosettes I) If m is an odd number, then*

$$(2.29) \quad L_{R_m}^2 = 4\pi m \tilde{A}_{R_m} - 8\pi m \tilde{A}_{E_{0.5,m}(R_m)} - \pi m \tilde{A}_{CWMS(R_m)}.$$

If m is an even number, then

$$(2.30) \quad L_{R_m}^2 = -2\pi m \tilde{A}_{R_m} + 4\pi m \tilde{A}_{E_{0.5,m}(R_m)} + \frac{\pi m}{2} \tilde{A}_{CWMS(R_m)}.$$

Proof. Let m be odd. Then by (2.7) and (2.25) we get that

$$\begin{aligned} p_{E_{0.5,m}(R_m)}(\theta) &= \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right), \\ p_{CWMS(R_m)}(\theta) &= 2 \sum_{\substack{n=2, \\ n \text{ is even}}}^{\infty} \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right). \end{aligned}$$

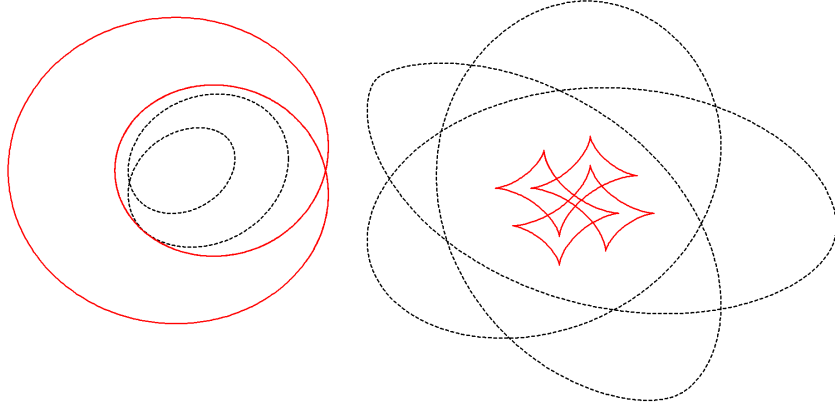


FIGURE 3. Two rosettes R_2 and R_3 (the dashed lines) and $\text{CWMS}(R_2), \text{CWMS}(R_3)$. Their support functions are $p_{R_2}(\theta) = 10 + 4 \cos \frac{\theta}{2} + \sin 2\theta$ and $p_{R_3}(\theta) = 12 + \cos \frac{2\theta}{3} + 4 \cos \frac{5\theta}{3} - \sin 2\theta$, respectively.

Therefore

$$\begin{aligned}\tilde{A}_{E_{0.5,m}(R_m)} &= -\frac{\pi}{4m} \sum_{\substack{n=2, \\ n \text{ is odd}}}^{\infty} (n^2 - m^2)(a_n^2 + b_n^2), \\ \tilde{A}_{\text{CWMS}(R_m)} &= -\frac{2\pi}{m} \sum_{\substack{n=2, \\ n \text{ is even}}}^{\infty} (n^2 - m^2)(a_n^2 + b_n^2).\end{aligned}$$

Let m be even. Then similarly one can get that

$$\begin{aligned}p_{E_{0.5,m}(R_m)}(\theta) &= a_0 + \sum_{\substack{n=2, \\ n \text{ is even}}}^{\infty} \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right), \\ p_{\text{CWMS}(R_m)}(\theta) &= -2a_0 + 2 \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} \left(a_n \cos \frac{n\theta}{m} + b_n \sin \frac{n\theta}{m} \right), \\ \tilde{A}_{E_{0.5,m}(R_m)} &= \frac{m\pi a_0^2}{2} - \frac{\pi}{4m} \sum_{\substack{n=2, \\ n \text{ is even}}}^{\infty} (n^2 - m^2)(a_n^2 + b_n^2), \\ \tilde{A}_{\text{CWMS}(R_m)} &= 8\pi m a_0^2 - \frac{2\pi}{m} \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} (n^2 - m^2)(a_n^2 + b_n^2).\end{aligned}$$

Then by (2.13) it is easy to verify that (2.29) and (2.30) hold. \square

We recall definition of an offset (this set is also known as a *parallel* set).

Definition 2.18. Let M be a regular positively oriented closed curve. An α -offset of M is the following set

$$F_{\alpha}(M) = \{a + \alpha \cdot \mathbf{n}(a) \mid a \in M\},$$

where $\mathbf{n}(a)$ is a continuous unit normal vector field to M at a compatible with the orientation of M .

Offset curves and surfaces are well-known geometric objects in the field of mathematics and computer aided geometric design, possibly because they give a powerful tool in many applications. It is well known that offsets of curves generically admit at most cusp singularities, singularities of all offsets of a curve form an evolute and set of all points of self-intersections of offsets forms a medial axis. For details see [10, 15, 21, 25, 31] and the literature therein.

In [38] we define the Spherical Measure Set for simply closed curves as follows.

Definition 2.19. The *Spherical Measure Set* of a regular positively oriented simply closed curve M is an offset at level $\frac{L_M}{2\pi}$,

$$(2.31) \quad \text{SMS}(M) = F_{\frac{L_M}{2\pi}}(M) = \left\{ a + \frac{L_M}{2\pi} \cdot \mathbf{n}(a) \mid a \in M \right\},$$

where \mathbf{n} is the unit normal vector field compatible with the orientation of M .

In this paper we generalize this notion for rosettes.

Definition 2.20. The *Spherical Measure Set* of an m -rosette R_m is an offset at level $\frac{L_{R_m}}{2m\pi}$,

$$(2.32) \quad \text{SMS}(R_m) = F_{\frac{L_{R_m}}{2m\pi}}(R_m) = \left\{ a + \frac{L_{R_m}}{2m\pi} \cdot \mathbf{n}(a) \mid a \in R_m \right\}.$$

It is easy to see that the support function of $\text{SMS}(R_m)$ is in the following form

$$(2.33) \quad p_{\text{SMS}(R_m)}(\theta) = p(\theta) - \frac{L_{R_m}}{2m\pi},$$

and has the following Fourier series

$$(2.34) \quad p_{\text{SMS}(R_m)}(\theta) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{m} + b_n \sin \frac{n\pi}{m} \right).$$

Directly by (2.34) we got the following propositions.

Proposition 2.21. A rosette R_m is a circle if and only if $\text{SMS}(R_m) = \{\Theta\}$.

Proposition 2.22. If m is odd and R_m is a rosette of constant width, then $\text{SMS}(R_m) = E_{0.5,m}(R_m)$.

Theorem 2.23. Let R_m be a generic rosette. Then the number of cusp singularities of $\text{SMS}(R_m)$ is even and not smaller than 2.

Proof. A curve $\text{SMS}(R_m)$ is singular if and only if

$$(2.35) \quad \rho_{\text{SMS}(R_m)}(\theta) = 0,$$

which is equivalent to

$$(2.36) \quad \rho(\theta) - \frac{L_{R_m}}{2m\pi} = 0.$$

The number of cusp singularities of $\text{SMS}(R_m)$ is equal to the number of zeros of $f(\theta) = \rho(\theta) - \frac{L_{R_m}}{2m\pi}$ in the range $[0, 2m\pi)$. Since $f(0) = f(2m\pi)$, the number of

zeros is even. Since $\int_0^{2m\pi} f(\theta)d\theta = L_{R_m} - L_{R_m} = 0$, there are at least two zeros of $f(\theta)$ in the range $[0, 2m\pi)$. □

In Fig. 4 we present examples of the SMS of a 2-rosette and a 3-rosette.

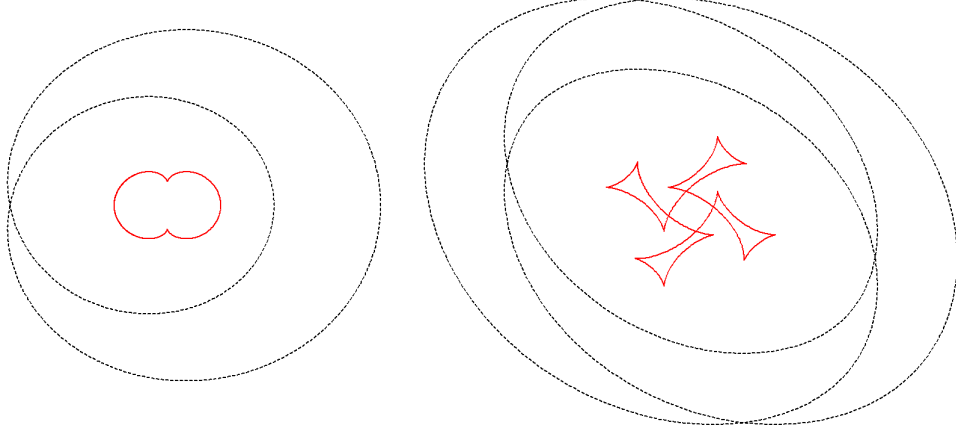


FIGURE 4. Rosettes R_2 and R_3 (the dashed lines) and $\text{SMS}(R_2), \text{SMS}(R_3)$. Support functions of R_2 and R_3 are equal to $p_{R_2}(\theta) = 3 + \cos \frac{\theta}{2}$ and $p_{R_3}(\theta) = 4 + \cos \frac{2\theta}{3} - \frac{1}{2} \sin 2\theta$, respectively.

Theorem 2.24. (*Isoperimetric equality for rosettes II*) *If m is odd and R_m is an m -rosette which is not of constant width, then*

$$(2.37) \quad L_{R_m}^2 = 4\pi m \tilde{A}_{R_m} + 4\pi m \tilde{A}_{\text{SMS}(R_m)}.$$

Proof. By (2.34) the oriented area of the Spherical Measure Set of R_m is in the following form

$$(2.38) \quad \begin{aligned} \tilde{A}_{\text{SMS}(R_m)} &= \frac{1}{2} \int_0^{2m\pi} \left(p_{\text{SMS}(R_m)}^2(\theta) - p_{\text{SMS}(R_m)}'^2(\theta) \right) d\theta \\ &= -\frac{\pi}{2m} \sum_{n=1}^{\infty} (n^2 - m^2)(a_n^2 + b_n^2). \end{aligned}$$

Then (2.37) is an easy consequence of (2.12), (2.13) and (2.38). □

3. AFFINE λ -EQUIDISTANTS OF THE UNION OF ROSETTES

We recall the definitions and theorems from [9] and then prove Theorem 3.12 and Theorem 3.15. We also simplify definitions and theorems from [9] to match only rosettes.

Definition 3.1. Let R_m be a generic m -rosette and let $f : S^1 \rightarrow \mathbb{R}^2$ be a parameterization of R_m in terms of $p(\theta), \theta$. Then the *sequence of parallel points* \mathcal{S} is the following sequence $(s_0, s_1, \dots, s_{2m-1})$, where $s_k = k\pi$.

Definition 3.2. The set of parallel arcs for an m -rosette is the following set

$$\Phi_0 = \{(0, 1), (1, 2), \dots, (2m - 2, 2m - 1), (2m - 1, 0)\}.$$

Proposition 3.3. ([9]) Let R_m be a generic m -rosette and let $f : S^1 \rightarrow \mathbb{R}^2$ be a parameterization of R_m in terms of $p(\theta), \theta$. For every two tuples $(k_1, l_1), (k_2, l_2)$ in Φ_0 the well defined map

$$f\left((s_{k_1}, s_{l_1})\right) \ni p \mapsto P(p) \in f\left((s_{k_2}, s_{l_2})\right),$$

such that $p, P(p)$ is a parallel pair of R_m , is a diffeomorphism.

Remark 3.4. The set of parallel arcs includes tuples of integers that are related to arcs of the m -rosette from which we can construct arcs of branches of an affine equidistants, see Fig. 5.

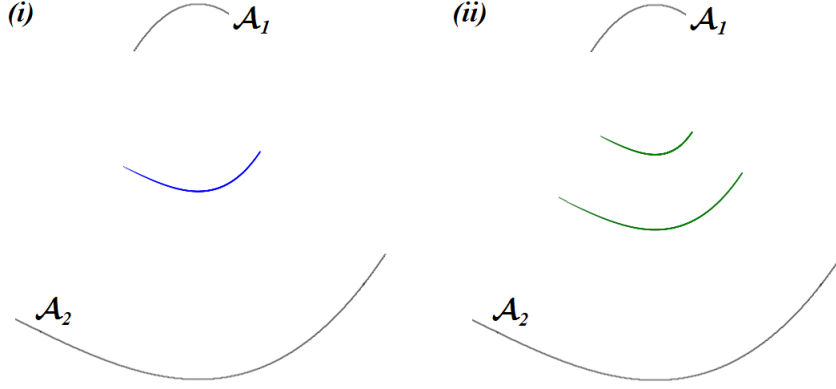


FIGURE 5. Two arcs $\mathcal{A}_1, \mathcal{A}_2$ which are related to the tuples in the set of parallel arcs and (i) $E_{\frac{1}{2}}(\mathcal{A}_1 \cup \mathcal{A}_2)$, (ii) $E_{\frac{2}{5}}(\mathcal{A}_1 \cup \mathcal{A}_2)$.

Definition 3.5. Let $(k_1, k_2), (l_1, l_2)$ belong to the set of parallel arcs, then $\frac{k_1}{l_1} \frac{k_2}{l_2}$ denotes the following set (an arc)

$$\left\{ \lambda a + (1 - \lambda)b \mid a \in f([s_{k_1}, s_{k_2}]), b \in f([s_{l_1}, s_{l_2}]), a, b \text{ is a parallel pair of } R_m \right\}$$

for an affine λ -equidistant for a fixed value of λ .

In addition $\frac{k_1}{l_1} \frac{k_2}{l_2} \dots \frac{k_{n-1}}{l_{n-1}} \frac{k_n}{l_n}$ denotes $\bigcup_{i=1}^{n-1} \frac{k_i}{l_i} \frac{k_{i+1}}{l_{i+1}}$. We will call this set a *glueing scheme* for $E_\lambda(R_m)$ for a fixed value of λ .

Definition 3.6. A *maximal glueing scheme* for $E_\lambda(R_m)$ for $\lambda \neq 0, 1$ is a glueing scheme which is a maximal element of the set of all glueing schemes for $E_\lambda(R_m)$ equipped with the inclusion relation.

Remark 3.7. In [9] we prove among other things that each maximal glueing scheme corresponds to a branch of an affine equidistant. If $\lambda \neq 0, \frac{1}{2}, 1$, then $E_\lambda(R_m)$ is the

union of $2 \binom{|\Phi_0|}{2}$ different arcs. If $\lambda = \frac{1}{2}$, then $E_{\frac{1}{2}}(R_m)$ is the union of $\binom{|\Phi_0|}{2}$ different arcs. Furthermore if $\lambda \neq 0, \frac{1}{2}, 1$, then the maximal glueing scheme of a branch of $E_\lambda(R_m)$ is in the form $\frac{k_1 \ k_2 \ \dots \ k_{n-1} \ k_n}{l_1 \ l_2 \ \dots \ l_{n-1} \ l_n}$, where $(k_1, l_1) = (k_n, l_n)$ and if $\lambda = \frac{1}{2}$, then the maximal glueing scheme of a branch of $E_\lambda(R_m)$ is in the form $\frac{k_1 \ k_2 \ \dots \ k_{n-1} \ k_n}{l_1 \ l_2 \ \dots \ l_{n-1} \ l_n}$, where $(k_1, l_1) = (k_n, l_n)$ or $(k_1, l_1) = (l_n, k_n)$.

Definition 3.8. Let M be a smooth regular curve. Let a, b be parallel pair of M and let us assume that the curvature at a and b does not vanish. We say that M is *curved in the same side* at a and b if the center of curvature of M at a and the center of curvature of $\tau_{a-b}(M)$ at $a = \tau_{a-b}(b)$ are lying on the same side of the tangent line to M at a .

Proposition 3.9. ([9]) *Let M be a smooth regular curve.*

- (i) *If M is curved in the same side at a parallel pair a, b , then $\lambda a + (1 - \lambda)b$ is a C^∞ regular point of $E_\lambda(M)$ for $\lambda \in (0, 1)$.*
- (ii) *If M is not curved in the same side at a parallel pair a, b , then $\lambda a + (1 - \lambda)b$ is a C^∞ regular point of $E_\lambda(M)$ for $\lambda \in (-\infty, 0) \cup (1, \infty)$.*

Theorem 6.1 in [9] and theorems in previous chapter (mainly Theorems 2.8, 2.9 and 2.17) show the geometry of the affine λ -equidistants and in particular of the Wigner caustic of rosettes in a fairly detailed way. To study the geometry of affine λ -equidistants for the union of two rosettes we want in particular study the geometry of branches which are created from both M and N . Therefore we present the following definition.

Definition 3.10. Let M and N be planar regular closed curves. Then an affine λ -equidistant of the pair M, N is the following set

$$E_\lambda(M, N) = \left\{ \lambda x + (1 - \lambda)y \mid (x \in M \wedge y \in N) \vee (x \in N \wedge y \in M), \right. \\ \left. x, y \text{ is a parallel pair of } M \cup N \right\}.$$

Note that $E_\lambda(M, M) = M \cup E_\lambda(M)$ and $E_\lambda(M \cup N) = E_\lambda(M) \cup E_\lambda(M, N) \cup E_\lambda(N)$.

Lemma 3.11. *Let M be a closed smooth curve with at most cusp singularities and let the rotation number of M be an integer. Then the number of cusp singularities of M is even.*

Proof. A continuous normal vector field to the germ of a curve with the cusp singularity is directed outside the cusp on the one of two connected regular components and is directed inside the cusp on the other component as it is showed in Fig. 6.

Since the rotation number of M is an integer, we get that the number of cusps of M is even. □

Let $\gcd(m, n)$ and $\text{lcm}(m, n)$ denote the greatest common divisor and the least common multiplier of two positive integers m and n , respectively.

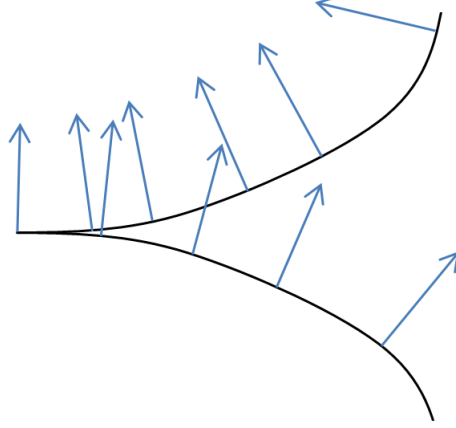


FIGURE 6. The cusp singularity with a continuous normal vector field. Vectors in upper regular component of a curve are directed outside the cusp, others are directed inside the cusp.

Theorem 3.12. *Let R_{m_1} and R_{m_2} be two generic rosettes. Then*

- (i) *there exist $2 \gcd(m_1, m_2)$ branches of $E_{0.5}(R_{m_1}, R_{m_2})$.*
- (ii) *at least $\gcd(m_1, m_2)$ branches of $E_{0.5}(R_{m_1}, R_{m_2})$ are rosettes.*
- (iii) *there exist $4 \gcd(m_1, m_2)$ branches of $E_\lambda(R_{m_1}, R_{m_2})$ for $\lambda \neq 0, \frac{1}{2}, 1$.*
- (iv) *at least $2 \gcd(m_1, m_2)$ branches of $E_\lambda(R_{m_1}, R_{m_2})$ for $\lambda \neq 0, \frac{1}{2}, 1$ are rosettes.*
- (v) *every branch of $E_\lambda(R_{m_1}, R_{m_2})$ for $\lambda \neq 0, 1$ has the rotation number equal to $\text{lcm}(m_1, m_2)$.*
- (vi) *the number of cusps of every branch of $E_\lambda(R_{m_1}, R_{m_2})$ for $\lambda \neq 0, 1$ is even.*

Proof. The sets of parallel arcs Φ, Φ' of rosettes R_{m_1}, R_{m_2} are as follows.

$$\begin{aligned}\Phi &= \{(0, 1), (1, 2), \dots, (2m_1 - 2, 2m_1 - 1), (2m_1 - 1, 0)\} \\ \Phi' &= \{(0', 1'), (1', 2'), \dots, (2m'_2 - 2, 2m'_2 - 1), (2m'_2 - 1, 0')\},\end{aligned}$$

where $a' := a + 2m_1$.

Therefore the set of parallel arcs of $R_{m_1} \cup R_{m_2}$ is in the following form.

$$(3.1) \quad \Phi_0 = \{(0, 1), (1, 2), \dots, (2m_1 - 2, 2m_1 - 1), (2m_1 - 1, 0), \\ (0', 1'), (1', 2'), \dots, (2m'_2 - 2, 2m'_2 - 1), (2m'_2 - 1, 0')\}.$$

Let $E_{\lambda, k}(R_{m_1}, R_{m_2})$ for an integer k denote a branch of $E_\lambda(R_{m_1}, R_{m_2})$.

Let us notice that the maximal glueing scheme of $E_{0.5, k}(R_{m_1}, R_{m_2})$ for $k = 0, 1, 2, \dots, 2 \gcd(m_1, m_2) - 1$ is in the following form

$$(3.2) \quad \begin{array}{ccccc} 0 & 1 & \dots & 2m_1 - 1 & 0 \\ k' & k' + 1 & \dots & k' - 1 & k' \end{array},$$

where on the upper part of (3.2) we have repetitively the following segment

$$\overline{0 \quad 1 \quad 2 \quad \dots \quad 2m_1 - 1 \quad 0}$$

of the length $2m_1$ and on the lower part of (3.2) we have repetitively the following segment

$$\overline{k' \quad k' + 1 \quad k' + 2 \quad \dots \quad k' - 1 \quad k'}$$

of the length $2m_2$. Therefore the maximal glueing scheme (3.2) is made of $2\text{lcm}(m_1, m_2)$ arcs.

The number of arcs of branches of $E_{0.5}(R_{m_1} \cup R_{m_2})$ is equal to $\binom{|\Phi_0|}{2}$. Because the number of arcs of branches of $E_{0.5}(R_{m_1})$ (resp. of $E_{0.5}(R_{m_2})$) is equal to $\binom{2m_1}{2}$ (resp. $\binom{2m_2}{2}$), therefore the number of arcs of $E_{0.5}(R_{m_1}, R_{m_2})$ is equal to $\binom{|\Phi_0|}{2} - \binom{2m_1}{2} - \binom{2m_2}{2} = 4m_1m_2$. Since we already used $2\text{gcd}(m_1, m_2) \cdot 2\text{lcm}(m_1, m_2) = 4m_1m_2$ arcs in maximal glueing schemes in the form (3.2) we cannot construct more branches of $E_{0.5}(R_{m_1}, R_{m_2})$.

Let $\lambda \neq 0, \frac{1}{2}, 1$. Using the same arguments as previous we can show that all maximal glueing schemes of $E_{\lambda,k}(R_{m_1}, R_{m_2})$ for $k = 0, 1, 2, \dots, 2\text{gcd}(m_1, m_2) - 1$ are in the form (3.3) and for $k = 2\text{gcd}(m_1, m_2), 2\text{gcd}(m_1, m_2) + 1, \dots, 4\text{gcd}(m_1, m_2) - 1$ are in the form (3.4), where $a'' := (a - 2\text{gcd}(m_1, m_2))'$.

$$(3.3) \quad \overline{\begin{array}{ccccc} 0 & 1 & \dots & 2m_1 - 1 & 0 \\ k' & k' + 1 & \dots & k' - 1 & k' \end{array}}$$

$$(3.4) \quad \overline{\begin{array}{ccccc} k'' & k'' + 1 & \dots & k'' - 1 & k'' \\ 0 & 1 & \dots & 2m_1 - 1 & 0 \end{array}}$$

where on the upper part of (3.3) (resp. the lower part of (3.4)) we have repetitively the following segment

$$\overline{0 \quad 1 \quad 2 \quad \dots \quad 2m_1 - 1 \quad 0}$$

of the length $2m_1$ and on the lower part of (3.3) (resp. the upper part of (3.4)) we have repetitively the following segment

$$\overline{\begin{array}{cccccc} k' & k' + 1 & k' + 2 & \dots & k' - 1 & k' \\ \text{(resp. } k'' & k'' + 1 & k'' + 2 & \dots & k'' - 1 & k'') \end{array}}$$

of the length $2m_2$.

By (3.2), (3.3) and (3.4) let us notice that every branch $E_{\lambda,k}(R_{m_1}, R_{m_2})$ is made of $2\text{lcm}(m_1, m_2)$ arcs and each arc has the rotation number equal to $\frac{1}{2}$. Since the rotation number of every branch $E_{\lambda,k}(R_{m_1}, R_{m_2})$ is an integer and for generic rosettes R_{m_1}, R_{m_2} every branch $E_{\lambda,k}(R_{m_1}, R_{m_2})$ are smooth curves with at most cusp singularities, we get that the number of cusp singularities of every branch $E_{\lambda,k}(R_{m_1}, R_{m_2})$ is even (see Lemma 3.11).

Let $S^1 \ni s \mapsto f_i(s) \in \mathbb{R}$ be the parameterization of R_{m_i} for $i = 1, 2$. Without loss of generality let us assume that curve R_{m_1} at $f_1(0)$ and R_{m_2} at $f_2(0')$ are curved in the same direction. Then one can see that branches $E_{0.5,2l}(R_{m_1}, R_{m_2})$ for $l = 0, 1, \dots, \text{gcd}(m_1, m_2) - 1$ have the property that for any parallel pair a_1, a_2 such that $a_1 \in R_{m_1}, a_2 \in R_{m_2}$ and the point $\frac{a_1 + a_2}{2}$ belongs to one of the branches above, the curves R_{m_1}, R_{m_2} are curved in the same side, then by Proposition 3.9

these branches are rosettes. Using the same method one can show that at least $2 \gcd(m_1, m_2)$ branches of $E_\lambda(R_{m_1}, R_{m_2})$ for $\lambda \neq 0, \frac{1}{2}, 1$ are rosettes. \square

Remark 3.13. The set $E_\lambda(R_{m_1}, R_{m_2})$ can be the union of only rosettes (see Fig. 7).

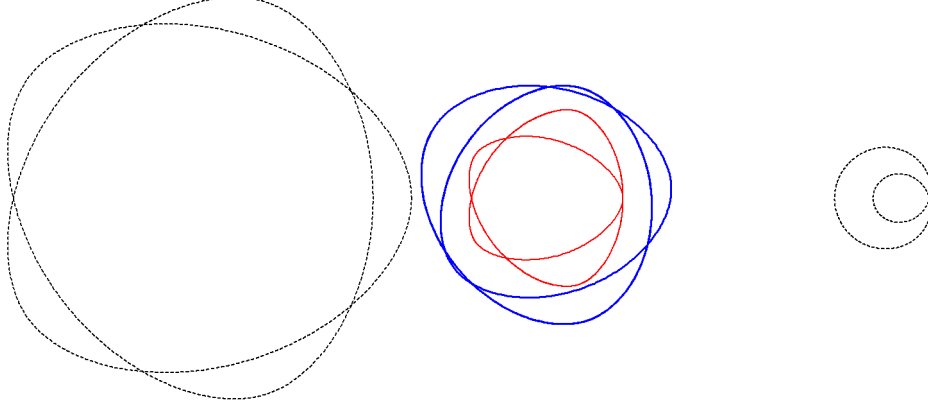


FIGURE 7. Two rosettes R_2, R'_2 (the dashed lines) and two branches of $E_{0.5}(R_2, R'_2)$ (the normal and the bold line). The support functions of R_2 (resp. R'_2) is $p_{R_2}(\theta) = 10 + \cos \frac{5\theta}{2}$ (resp. $p_{R'_2}(\theta) = 2 + 36 \cos \theta + \sin \frac{\theta}{2}$).

By the maximal glueing schemes (3.2), (3.3) and (3.4) we can get the following lemma.

Lemma 3.14. *If $\lambda = \frac{1}{2}$ then the $2\pi \text{lcm}(m_1, m_2)$ -periodic support function of a branch $E_{0.5,k}(R_{m_1}, R_{m_2})$ for $k = 0, 1, 2, \dots, 2 \gcd(m_1, m_2) - 1$ is in the following form*

$$(3.5) \quad p_{E_{0.5,k}(R_{m_1}, R_{m_2})}(\theta) = \frac{1}{2} \left(p_{R_{m_1}}(\theta) + (-1)^k p_{R_{m_2}}(\theta + k\pi) \right).$$

If $\lambda \neq 0, \frac{1}{2}, 1$ then the $2\pi \text{lcm}(m_1, m_2)$ -periodic support function of a branch $E_{\lambda,k}(R_{m_1}, R_{m_2})$ for $k = 0, 1, 2, \dots, 2 \gcd(m_1, m_2) - 1$ (resp. for $k = 2 \gcd(m_1, m_2), 2 \gcd(m_1, m_2) + 1, \dots, 4 \gcd(m_1, m_2) - 1$) is in the form (3.6) (resp. in the form (3.7)).

(3.6)

$$p_{E_{\lambda,k}(R_{m_1}, R_{m_2})}(\theta) = \lambda p_{R_{m_1}}(\theta) + (-1)^k (1 - \lambda) p_{R_{m_2}}(\theta + k\pi),$$

(3.7)

$$p_{E_{\lambda,k}(R_{m_1}, R_{m_2})}(\theta) = (1 - \lambda) p_{R_{m_1}}(\theta) + (-1)^k \lambda p_{R_{m_2}}(\theta + (k - 2 \gcd(m_1, m_2))\pi).$$

Theorem 3.15. (i) *If k is even, $k < 2 \gcd(m_1, m_2)$ and $\lambda \in (0, 1)$, then*

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} = \lambda \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + (1 - \lambda) \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(ii) If k is even, $k \geq 2 \gcd(m_1, m_2)$ and $\lambda \in (0, 1) - \left\{\frac{1}{2}\right\}$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} = (1 - \lambda) \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + \lambda \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(iii) If k is odd, $k < 2 \gcd(m_1, m_2)$ and $\lambda \in (0, 1)$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} \leq \lambda \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + (1 - \lambda) \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(iv) If k is odd, $k \geq 2 \gcd(m_1, m_2)$ and $\lambda \in (0, 1) - \left\{\frac{1}{2}\right\}$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} \leq (1 - \lambda) \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + \lambda \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(v) If k is odd, $k < 2 \gcd(m_1, m_2)$ and $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} = |\lambda| \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + |1 - \lambda| \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(vi) If k is odd, $k \geq 2 \gcd(m_1, m_2)$ and $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} = |1 - \lambda| \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + |\lambda| \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(vii) If k is even, $k < 2 \gcd(m_1, m_2)$ and $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} \leq |\lambda| \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + |1 - \lambda| \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

(viii) If k is even, $k \geq 2 \gcd(m_1, m_2)$ and $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} \leq |1 - \lambda| \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + |\lambda| \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}.$$

Proof. We prove (i), other points are analogous. So let k be an even number and $k < 2 \gcd(m_1, m_2)$ and let $\lambda \in (0, 1)$. Therefore we get

$$\begin{aligned} L_{E_{\lambda,k}(R_{m_1}, R_{m_2})} &= \int_0^{2\pi \text{lcm}(m_1, m_2)} \rho_{E_{\lambda,k}(R_{m_1}, R_{m_2})}(\theta) d\theta \\ &= \int_0^{2\pi \text{lcm}(m_1, m_2)} p_{E_{\lambda,k}(R_{m_1}, R_{m_2})}(\theta) d\theta \\ &= \lambda \int_0^{2\pi \text{lcm}(m_1, m_2)} p_{R_{m_1}}(\theta) d\theta + (1 - \lambda) \int_0^{2\pi \text{lcm}(m_1, m_2)} p_{R_{m_2}}(\theta) d\theta \\ &= \lambda \frac{\text{lcm}(m_1, m_2)}{m_1} L_{R_{m_1}} + (1 - \lambda) \frac{\text{lcm}(m_1, m_2)}{m_2} L_{R_{m_2}}. \end{aligned}$$

□

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FACULTY OF MATHEMATICS AND INFORMATION SCIENCE, WARSAW UNIVERSITY OF TECHNOLOGY, UL. KOSZYKOWA 75,, 00-662 WARSZAWA

E-mail address: `zwierzynskim@mini.pw.edu.pl`